Math 210C Lecture 6 Notes

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1 General Nullstellensatz, Pullbacks, and Krull Dimension

1.1 Nullstellensatz for $\mathrm{Spec}(R)$

Let R be a commutative ring.If $T \subseteq R$, we defined $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(R) : T \subseteq \mathfrak{p}\}$. If $Y \subseteq \mathrm{Spec}(R)$, we defined $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. We defined the Zariski topology on $\mathrm{Spec}(R)$, where closet sets are the $V(T)$ or equivalently, $V(I)$, where $I \subseteq R$ is an ideal. Note that not all points are closed in $\mathrm{Spec}(R)$, just the maximal ideals.[1]

Proposition 1.1. *I and V provide mutually inverse, inclusion-reversing bijections between $\{$radical ideals of $R\} \leftrightarrow \{$closed sets in $\mathrm{Spec}(R)\}$. In fact, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for all ideals $\mathfrak{a} \subseteq R$, and $V(I)Y) = \overline{Y}$, where Y is the Zariski closure of Y.*

Proof. Note that

$$I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \sqrt{\mathfrak{a}},$$

where the second equality comes from the fact that in $R/\mathfrak{a}$, $\sqrt{0} = \bigcap_{\mathfrak{p}} \mathfrak{p} = I(V(0))$. We also have

$$V(I(Y)) = \{\mathfrak{p} \supseteq \bigcap_{\mathfrak{q} \in Y} \mathfrak{q}\} = \{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{q} \text{ for some } q \in Y\} \supseteq \overline{Y}$$

Also, $V(I(Y)) \subseteq V(I(\overline{Y})) = \overline{Y}$. $\square$

Corollary 1.1. *The Zariski closure of $Y \subseteq \mathrm{Spec}(R)$ is the set of prime ideals containing some element of Y.*

1.2 Pullbacks in the Zariski topology

Here is how our two notions of V and I relate.

Proposition 1.2. *Let $R = K[x_1, \dots, x_n]$, where K is algebraically closed. Let V' be the vanishing locus and I' be the ideal of vanishing in our original sense.*

[1] This topology is not even T_1.

1. *$\iota : \mathbb{A}_K^n \to \mathrm{Spec}(R)$ sending $(a_1, \ldots, a_n) \mapsto (x_1 - a_1, \ldots, x_n - a_n)$ is injective. The Zariski topology on $\mathbb{A}_K^n$ is the subspace topology for the Zariski topology on $\mathrm{Spec}(R)$.*

2. *If $\mathfrak{a} \subseteq R$ is an ideal, then $V(\mathfrak{a}) = V(\mathfrak{a}) \cap \mathbb{A}_K^n$.*

3. *If $Y \subseteq \mathrm{Spec}(R)$ is Zariski closed, then $I(Y) = I'(Y \cap \mathbb{A}_K^n)$.*

Definition 1.1. Let $a \in R$. $U_a := \mathrm{Spec}(R) \setminus V((a))$ is called a **principal open set** in $\mathrm{Spec}(R)$.

Proposition 1.3. *The U_a form a base for the Zariski topology on $\mathrm{Spec}(R)$.*

Proof. Let $U \subseteq \mathrm{Spec}(R)$ be open. Then $U = \mathrm{Spec}(R) \setminus V(I)$, where I is an ideal. Then $V(I) = \bigcap_{a \in I} U_a$, so $U = \bigcup_{a \in I} U_a$. □

Lemma 1.1. *Let $\varphi : R \to S$ be a ring homomorphism.*

1. *If $\mathfrak{q}$ is a prime ideal of S, then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R.*

2. *Suppose that φ is surjective and $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \supseteq \ker(\varphi)$. Then $\varphi(\mathfrak{p})$ is a prime ideal.*

Definition 1.2. Let $\varphi : R \to S$ be a ring homomorphism. The **pullback map** $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is $\varphi^*(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for all prime $\mathfrak{q} \subseteq S$.

Lemma 1.2. *φ^* is continuous.*

Proof. Let $\mathfrak{a} \subseteq R$ be an ideal. Then

$$(\varphi^*)^{-1}(V(\mathfrak{a})) = \{\mathfrak{q} \in \mathrm{Spec}(S) : \varphi^{-1}(\mathfrak{q}) \supseteq \mathfrak{a}\} = V(\varphi(\mathfrak{a})).$$ □

Remark 1.1. $R \mapsto \mathrm{Spec}(R)$ and $\varphi \mapsto \varphi^*$ defines a contravariant functor Ring $\to$ Top.

Example 1.1. Let $\pi : R \to F$ be a surjective ring homomorphism, where Fsi a field. $\pi^*((0)) = V(\ker(\pi)) = \{\ker(\pi)\}$, since $\ker(\pi)$ is maximal.

Example 1.2. If $\mathfrak{p}$ is prime, let $\varphi : R \to R_\mathfrak{p}$ be the natural map. Then $\mathrm{Spec}(R_\mathfrak{p}) = \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \text{ prime of } R, \mathfrak{q} \subseteq \mathfrak{p}\}$. So $\varphi^*(\mathfrak{q}R_\mathfrak{p}) = \mathfrak{q}$.

Example 1.3. Let $\varphi : \mathbb{C}[x] \to \mathbb{C}[x]$ be $\varphi(x) = x^2$. Then $\varphi^*((0)) = (0)$, and $\varphi^*((x-a)) = \{g \in \mathbb{C}[x] : g(a^2) = 0\} = (x - a^2)$. Since $\varphi^*((x-a)) = \varphi^*((x+a))$, this is a 2 to 1 map, aside from $a = 0$. What does $\mathrm{Spec}(\mathbb{C}[x])$ look like? It looks like $\mathbb{C} \cup \{(0)\}$

1.3 Krull dimension

Definition 1.3. The **length** of an strictly ascending chain of distinct prime ideals $(\mathfrak{p}_i)_{i=0}^n$ of R is n.

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

Example 1.4. Look at $R[x_1, \ldots, x_n]$, where R is a domain.

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n)$$

has length n.

Definition 1.4. The **Krull dimension** of R is the maximum of the lengths of all chains of prime ideals in $\operatorname{Spec}(R)$, if finite. Otherwise, we say the Krull dimension is infinite.

Example 1.5. If R is a field and $n \geq 1$, then $F[x]/(x^n)$ has Krull dimension 0.

Example 1.6. $\dim(\mathbb{Z}) = 1$. In fact, $\dim(R) = 1$ for all PIDs.

Example 1.7. Let F be a field. $F[x_1, x_2, \ldots]$ has infinite Krull dimension because of the chain

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$$

Lemma 1.3. *If $\pi : R \to S$ is surjective, then $\dim(R) \geq \dim(S)$.*

Proposition 1.4. *If R is a domain, then $\dim(R[x]) = \dim(R) + 1$.*

Corollary 1.2. *Let $R = F[x_1, \ldots, x_n]$, where F is a field. Then $\dim(R) = n$.*